Exam Physics of Fluids, 02-07-2013

$$\frac{d}{dt}\int_{V(t)} F(\underline{x},t)\,dV = \int_{V(t)} \frac{\partial F}{\partial t}\,dV + \int_{A(t)} F(\underline{x},t)\,\underline{u}\cdot\underline{n}\,dA$$

(Reynolds' Transport Theorem)

① a

$$0 = \frac{d}{dt}\int_{V(t)} \rho(\underline{x},t)\,dV = \int_{V(t)} \frac{\partial \rho}{\partial t}\,dV + \int_{A(t)} \rho(\underline{x},t)\,\underline{u}\cdot\underline{n}\,dA$$

$$= \int_{V(t)} \frac{\partial \rho}{\partial t}\,dV + \int_{V(t)} \underline{\nabla}\cdot\left(\rho(\underline{x},t)\,\underline{u}\right)dV = \int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot\left(\rho(\underline{x},t)\,\underline{u}\right)\right]dV$$

To hold, integrand must vanish at all points:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot\left(\rho(\underline{x},t)\,\underline{u}\right) = 0$$

$$\rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho u_i\right) = 0$$

$$\frac{d}{dt}\int_{V(t)} \rho\,\underline{u}\,dV = \int_{V(t)} \rho\,\underline{g}\,dV + \int \underline{f}\,dA$$

$$= \int_{V(t)} \rho\,\underline{g}\,dV + \int_{A(t)} \underline{\underline{\tau}}\cdot\underline{n}\,dA$$

$$= \int_{V(t)} \left[\rho\,\underline{g} + \underline{\nabla}\cdot\underline{\underline{\tau}}\right]dA$$

$$\frac{d}{dt}\int_{V(t)} \rho \underline{u}\, dV = \int_{V(t)} \frac{\partial(\rho \underline{u})}{\partial t}\, dV + \int_{A(t)} \rho \underline{u}(\underline{u}\cdot\underline{n})\, dA$$

$$= \int_{V(t)} \frac{\partial(\rho \underline{u})}{\partial t}\, dV + \int_{V(t)} \underline{\nabla}\cdot\left(\rho \underline{u}\,\underline{u}\right) dV$$

$$= \int_{V(t)} \frac{\partial(\rho \underline{u})}{\partial t} + \underline{\nabla}\cdot\left(\rho \underline{u}\right)\cdot\underline{u}\; dV$$

~~This must hold for all points, for all t.~~

$$\cancel{\rightarrow \frac{\partial(\rho \underline{u})}{\partial t} + \underline{\nabla}(\rho \underline{u})\underline{u} = \rho g + \underline{\nabla}\cdot\underline{\underline{\tau}}}$$

This must hold $\forall x_i$, $\forall t$, so integral sign disappears.

$$\frac{\partial(\rho \underline{u})}{\partial t} + \nabla\cdot\left[(\rho\underline{u})\underline{u}\right] = \underline{u}\frac{\partial\rho}{\partial t} + \rho\frac{\partial \underline{u}}{\partial t} + \underline{u}\left(\nabla\cdot(\rho\underline{u})\right) + \rho\underline{u}(\nabla\cdot\underline{u})$$

$$= \rho\left(\frac{\partial\underline{u}}{\partial t} + u\cdot\underline{\nabla}\,\underline{u}\right) + \underline{u}\Big(\underbrace{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{u})}_{\hookrightarrow \text{ conservation of mass } \Rightarrow\, = 0}\Big)$$

$$= \rho\frac{D\underline{u}}{Dt}$$

$$\rightarrow \rho\frac{D\underline{u}}{Dt} = \rho g + \underline{\nabla}\cdot\underline{\underline{\tau}}$$

$$\Rightarrow \rho\frac{Du_i}{Dt} = \rho g_i + \frac{\partial}{\partial x_i}\tau_{ij}$$

b) $\underline{u}^* = \underline{u}/U$, $\underline{x}^* = \underline{x}/L$ $t^* = \Omega t$ $\boxed{p^* = \frac{p - p_0}{\rho U^2}}$, $g^* = g/g$

$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho g + \mu \nabla^2 \underline{u}$$

$$\rightarrow \rho \frac{\partial \underline{u}}{\partial t} + \rho\, \underline{u}\, \underline{\nabla} \cdot \underline{u} = -\nabla p + \rho g + \mu \nabla^2 \underline{u}$$

$$\Omega \rho U \frac{\partial u^*}{\partial t} + \underbrace{\frac{\rho U^2}{L}}_{\underline{u}^* \underline{\nabla}^* \cdot \underline{u}^*} = -\frac{\rho U^2}{L} \nabla^* p^* + \rho g g^* + \frac{\mu U}{L^2} \nabla^2 \underline{u}^* \Big| : \rho U$$

$$\Omega \frac{\partial u}{\partial t} + \frac{U}{L} \underline{u}^* \underline{\nabla}^* \cdot \underline{u}^* = -\frac{U}{L} \nabla^* p^* + \frac{g}{U} g^* + \frac{\mu}{\rho U L} \frac{U}{L} \nabla^{*2} \underline{u}^* \Big| \cdot L/U$$

$$\frac{\Omega L}{U} \frac{\partial u}{\partial t} + \underline{u}^* \underline{\nabla}^* \cdot \underline{u}^* = -\nabla^* p^* + \frac{gL}{U^2} g^* + \frac{\mu}{\rho U L} \nabla^{*2} u^*$$

$$\boxed{St \frac{\partial u}{\partial t} + \underline{u}^* \underline{\nabla}^* \cdot \underline{u}^* = -\nabla^* p^* + Fr^{-2} g^* + Re^{-1} \nabla^{*2} \underline{u}^*}$$

c)

	U	ρ	L	μ	Ω	g
M	0	1	0	1	0	0
L	1	-3	1	-1	0	1
T	-1	0	0	-1	-1	-2

$\Rightarrow$ $Re = \frac{\rho U L}{\mu}$

$Fr = \frac{U}{\sqrt{gL}}$

$St = \frac{\Omega L}{U}$

rank = 3

$\rightarrow$ 6-3 = 3 dimless parameters.

2) a.

	Δp	M	D	u	d	ρ	g
M	1	0	0	0	0	1	0
L	-1	1	1	1	1	-3	1
T	-2	0	0	-1	0	0	-2

4 dimless groups.

$\Pi_3 = D/M \qquad \Pi_4 = D/d$

$\Pi_1 = \Delta p \, \rho^\alpha u^\beta D^\gamma$

$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\alpha = -1, \quad \beta = -2, \quad 1 = 3 - 2 + \gamma \rightarrow \gamma = 0.$

$\Pi_1 = \dfrac{\Delta p}{\rho u^2},$

$\Pi_2 = g u^\alpha M^\beta$

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \alpha = -2, \ \beta = 1$$

$\Pi_2 = \dfrac{gM}{u^2}$

$\rightarrow$ equivalent to $St = \dfrac{u}{\sqrt{gD}}$

$\Pi_1 = \dfrac{\Delta p}{\rho u^2}, \quad \Pi_2 = \dfrac{gM}{u^2} \quad \Pi_3 = \dfrac{D}{M}, \quad \Pi_4 = \dfrac{D}{d}$

$$\rightarrow \frac{\Delta p}{\rho U^2} = f\left(\frac{gH}{U^2}, \frac{D}{H}, \frac{d}{D}\right).$$

b) Conservation of mass:

$$\cancel{\rho}\int_{(1)} U_1\, dA = \cancel{\rho}\int_{(2)} U_2\, dA \leftrightarrow \cancel{\pi} R_1^2 U_1 = \cancel{\pi} R_2^2 U_2$$

$$U_2 = \frac{R_1^2}{R_2^2} U_1 = 64\, U_1 = 64 \text{ cm/s}$$

$$Q = \cancel{\rho}\int_{(2)} U_2\, dA = \cancel{\rho}\pi R_2^2 U_2 = \cancel{3{,}14\pi}\cdot(1\cdot 10^{-3})^2 \cdot 0{,}64 = 0{,}64\cdot 10^{-3}\ m^3/s$$

c) Bernoulli: $\Delta p = p_2 - p_1$

$$\frac{1}{2}U_1^2 + g z_1 + p_1/\rho = \frac{1}{2}U_2^2 + g z_2 + p_2/\rho$$

$$\frac{1}{2}(U_1^2 - U_2^2) + g(z_1 - z_2) = \frac{\Delta p}{\rho}$$

$$\Delta p = \underbrace{\rho}_{1000}\left(\frac{1}{2}\left(10^{-4} - 4096\cdot 10^{-4}\right) - \underbrace{g}_{10}\underbrace{H}_{2\cdot 10^{-2}}\right) = 1000\left(-(2047{,}5)\cdot 10^{-4} - 2\cdot 10^{-1}\right)\frac{N}{m^2}$$

$$= -2{,}05 - 200\ \frac{N}{m^2} = -202\ \frac{N}{m^2}$$

d) $p_1 = p_2 - \Delta p = p_{atm} - \Delta p = 1{,}013\cdot 10^5 + 2{,}02\cdot 10^2 = 1{,}015\cdot 10^5\ \frac{N}{m^2}$

$$F = \int p\, dA = \pi R_1^2 p_1 = \pi\cdot(8\cdot 10^{-3})^2\cdot 1{,}015\cdot 10^5 = \pi\cdot 64\cdot 1{,}015\cdot 10^{-1}$$

$$= 6{,}5\pi = 18{,}7\ N$$

e) $\Delta p = \frac{\rho}{2}\left(u_1^2 - u_2^2\right) + \rho g (z_1 - z_2)$

$$= \frac{\rho}{2}\left(u_1^2 - \left(\frac{D}{d}\right)^2 u_2^2\right) - \rho g H$$

$$= \frac{\rho u^2}{2}\left(1 - \left(\frac{D}{d}\right)^2\right) - \rho g u$$

$$\frac{\Delta p}{\rho u^2} = \frac{1}{2}\left(1 - \left(\frac{D}{d}\right)^2\right) - \frac{\cancel{\rho} g H}{\cancel{\rho} u^2}$$

$$\Pi_1 = \frac{1}{2}\left(1 - \Pi_4^{\,2}\right) - \Pi_2$$

f) $\Pi_3 = \frac{D}{H} = \frac{16}{20} = \frac{4}{5} = 0{,}8 = \frac{D_m}{H_m}$

$$H_m = \frac{D_m}{0{,}8} = \frac{0{,}08}{0{,}8} = 0{,}1 \text{ m}$$

$$\Pi_4 = \frac{D}{d} = \frac{16}{2} = 8 = \frac{D_m}{d_m}$$

$$d_m = \frac{D_m}{8} = \frac{0{,}08}{8} = 0{,}01 \text{ m}$$

$$\Pi_2 = \frac{gH}{u^2} = \frac{10 \cdot 0{,}016}{1 \cdot 10^{-4}} = 1{,}6 \cdot 10^3 = \frac{g H_m}{u_m^2}$$

$$u_m = \sqrt{g H_m / 1{,}6 \cdot 10^3} = \sqrt{\frac{0{,}1}{1{,}6 \cdot 10^2}} = \sqrt{1/16} = 1/4 \text{ m/s}$$

g) $\Pi_1 = \frac{202}{1000 \cdot 10^{-4}} = 2020$. $\Delta p = 2020 \cdot \rho u_m^2 = 126125 \frac{N}{m^2}$

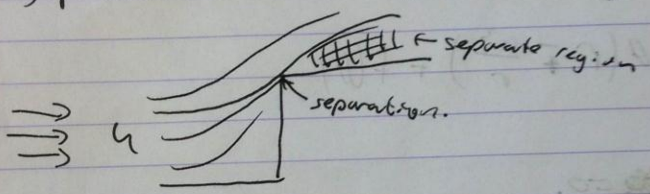
③ a) $Re = \frac{2\rho U a}{\mu} = \frac{1{,}23 \cdot \frac{\cancel{100}}{3{,}6} \cdot 6 \cdot 10^2}{1{,}81 \cdot 10^{-5}} = \frac{1{,}23 \cdot 6}{3{,}6} \cdot 10^7 \approx 2 \cdot 10^7$

No

b) Book pg 199

1) formation of viscous boundary layers containing rotational fluid: the thickness of these boundary layers, where viscous diffusion of vorticity is important, approaches zero for $Re \to \infty$.

2) possible formation of separate flow regions in real flow:



c) Horizontal flow and a doublet.

d) $x = r\cos\theta$, $r^2 = x^2 + y^2$
$y = r\sin\theta$ $\theta = \operatorname{atan} y/x$

$\phi = Ur\cos\theta + \frac{Ua^2\cos\theta}{r} = U\cos\theta\left(r + \frac{a^2}{r}\right)$

$u = \frac{\partial\phi}{\partial x}$, $v = \frac{\partial\phi}{\partial y}$, $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$

$\cancel{u = \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial x} = \frac{\partial\phi}{\partial r}\Big/\frac{\partial x}{\partial r} + \frac{\partial\phi}{\partial\theta}\Big/\frac{\partial x}{\partial\theta}}$

$\cancel{= U\cos\theta\left(1 + \frac{a^2}{r^2}\right)\Big/\cos\theta + U\sin\theta\left(r - \frac{a^2}{r}\right)\Big/r\sin\theta}$

$= 2U$

$$u_r = \frac{\partial \phi}{\partial r} = U\cos\theta\left(1 - \frac{a^2}{r^2}\right) = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$\frac{\partial \psi}{\partial \theta} = U\cos\theta\left(r - \frac{a^2}{r}\right)$$

$$\rightarrow \psi(\theta) = U\sin\theta\left(r - \frac{a^2}{r}\right) + f(r)$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\sin\theta\left(1 + \frac{a^2}{r^2}\right) = -\frac{\partial \psi}{\partial r}$$

$$\rightarrow \psi(r) = U\sin\theta\left(r - \frac{a^2}{r}\right) + f(\theta)$$

$$\psi(r) = \psi(\theta) \Rightarrow f(r) = f(\theta) = 0.$$

e) $u_r = U\cos\left(1 - \frac{a^2}{r^2}\right) \rightarrow u_r = 0 \ @ \ r = \pm a$

$u_\theta = -U\sin\theta\left(1 + \frac{a^2}{r^2}\right) \rightarrow u_\theta = 0 \ @ \ \theta = 0, \pi$

$\rightarrow$ stagnation points ; $(r, \theta) \Rightarrow (a, 0), (a, \pi)$

f) $\psi(r, \theta) = U\left(r - \frac{a^2}{r}\right)\sin\theta$

$\psi(a, 0) = \psi(a, \pi) = 0$

g) Bernoulli: $\frac{1}{2}U^2 + gz + \frac{p}{\rho} = cst$

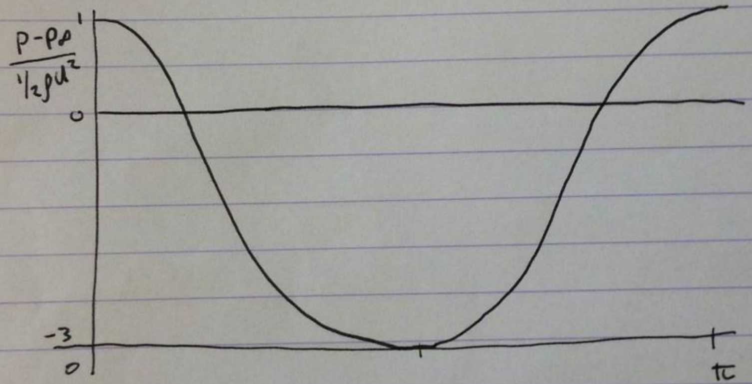
$$\frac{1}{2}U_\infty^2 + \frac{p_0}{\rho} = \frac{p}{\rho} \rightarrow p = \frac{\rho}{2}U_\infty + p_0$$

$$= \frac{1{,}23}{2}\cdot\left(\frac{100}{3{,}6}\right)^2 + 101{,}3\cdot10^3$$

$$= \cancel{0{,}615}\cdot\cancel{} + 101{,}3\cdot10^3 = \cancel{}\cancel{+101{,}3\cdot10^3}$$

$$= 474{,}5 + 101{,}3\cdot10^3 = 101{,}8\cdot10^3 \text{ Pa}$$

b)



d'Alembert's paradox: blz 208, onderaan.